Projectile Motion: Special & Important Example of two-dimensional motion.

"A particle moving in the vertical plane with some initial velocity whose acceleration is always free-fall acceleration (g). Such particle is called a projectile."

The motion of projectile is called projectile motion.

Basic Assumption: \( \rightarrow \) No air effect.

2D Projectile Motion:

The projectile is launched with an initial velocity \( \vec{V}_0 \) so that:

\[
\begin{align*}
\vec{V}_0 &= V_{0x} \hat{i} + V_{0y} \hat{j} \\
\text{Assume the angle } \theta_0 \text{ would be } V_{0y} \frac{V_0}{ \cos \theta_0} \text{ and the x-direction.}
\end{align*}
\]

So,

\[
\begin{align*}
V_{0x} &= V_0 \cos \theta_0 \\
V_{0y} &= V_0 \sin \theta_0
\end{align*}
\]

During the motion, the projectile's position vector \( \vec{r} \) and velocity \( \vec{v} \) change continuously, but its \( \vec{a} \) is a constant.
and always directed vertically downward.

The projectile has zero horizontal acceleration.

In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

Along the whole motion, \( V_{0x} \) would be constant, only \( V_{0y} \) changes with time.

Now, tracking the motion. Before that, recalling:

\[
\begin{align*}
    v_f &= v_i + at \\
    s &= v_i t + \frac{1}{2} at^2 \\
    2as &= v_f^2 - v_i^2
\end{align*}
\]
Horizontal Motion: As there is no horizontal acceleration throughout the motion, therefore its velocity's x-component remains same.

\[ S = Vt + \frac{1}{2}at^2 \]

\[ x - x_0 = V_{0x}t \Rightarrow x - x_0 = (V_0 \cos \theta_0) t \]

Vertical Motion: \[ a = -g = -9.8 \text{ m/s}^2 \]

Remember \[ g \neq -9.8 \text{ m/s}^2 \]

\[ S = Vt - \frac{1}{2}gt^2 \]

\[ y - y_0 = V_{0y}t - \frac{1}{2}gt^2 \Rightarrow y - y_0 = (V_0 \sin \theta_0) t - \frac{1}{2}gt^2 \]

Other Forms becomes:

\[ V_y = V_0 \sin \theta_0 - gt \quad (i.e. V_y = V_0 + at) \]

\[ -2g(y - y_0) = V_y^2 - (V_0 \sin \theta_0)^2 \]

\[ h - v^2 = v_0^2 - v_i^2 \]
Trajectory or Equation of the Path

Path through Space "ONLY."

Can be done by eliminating time from eqn 1 and 2.

\[ x - x_0 = V_0 \cos \theta_0 t \]
\[ t = \frac{x - x_0}{V_0 \cos \theta_0} \]

Put in eqn 2:

\[ y - y_0 = V_0 \sin \theta_0 \left( \frac{x - x_0}{V_0 \cos \theta_0} \right) - \frac{1}{2} g \left( \frac{x - x_0}{V_0 \cos \theta_0} \right)^2 \]

For the sake of easiness, put \( y_0 = x_0 = 0 \)

\[ y = (\tan \theta_0) x - \frac{1}{2} \frac{g x^2}{(V_0 \cos \theta_0)^2} \]

\[ y = ax + bx^2 \]  

Equation of Parabola.
The Horizontal Range

"The horizontal distance the projectile has to travel to return to its initial height (the height at which it is launched)."

To find range \( R \), put \( x-x_0 = R \) and \( y-y_0 = 0 \)

Eqs (1) and (2) becomes:

\[ R = (V_0 \cos \theta_0) t \quad \text{and} \quad 0 = V_0 \sin \theta_0 \cdot t - \frac{1}{2} gt^2 \]

Eliminating \( t \), we get:

\[ R = \frac{2V_0^2 \sin \theta_0 \cos \theta_0}{g} \]

\[ R = \frac{V_0^2 \sin 2\theta_0}{g} \]

\[ \sin 2\theta_0 = 2 \sin \theta_0 \cos \theta_0 \]

\[ \varphi_{max} = \frac{V_0^2}{g} \]

\[ \sin 2\theta_0 = 1 \Rightarrow 2\theta_0 = 90^\circ \Rightarrow \theta_0 = 45^\circ \]
Maximum Height \( (H) \)

\[ x - x_0 = 0 \quad \text{and} \quad y - y_0 = H_{\text{max}} \]

Equations become modified.

To find \( H_{\text{max}} \), use 3rd equation as follows:

\[-2g H_{\text{max}} = (V_{fy})^2 - (V_0 \sin \theta_0)^2\]

\[-2g H_{\text{max}} = 0 - V_0^2 \sin^2 \theta_0\]

\[\Rightarrow H_{\text{max}} = \frac{V_0^2 \sin^2 \theta_0}{2g}\]

**Time of Flight: \( (t_f) \)**

("Time b/w the instants of launch & impact")

\[t_f = 2 \cdot t_{\text{height (max)}}\]

For \( t_{\text{max height}} \): \( V_{fy} = 0 \) therefore

\[V_{fy} = V_{iy} - g \cdot t_{\text{max height}}\]

\[t_{\text{height max}} = \frac{V_0 \sin \theta_0}{g}\]

\[\Rightarrow \theta \text{ becomes as}\]
\[ t_f = \frac{2V_0 \sin \Theta_0}{g} \]